Definition of velocity in doubly special relativity theories

Piotr Kosiński and Paweł Maślanka
Department of Theoretical Physics II, University of Łódź, Pomorska 149/153, 90-236 Łódź, Poland
(Received 16 June 2003; published 15 September 2003)

We discuss the definition of particle velocity in doubly special relativity theories. The general formula relating the velocity and four-momentum of the particle is given.

DOI: 10.1103/PhysRevD.68.067702 PACS number(s): 11.30.−j, 03.30.+p

I. INTRODUCTION

Recently there have appeared a number of papers [1–28] devoted to the so called doubly special relativity (DSR) theories. These are modifications of special relativity in which some particular value of energy/momentum joins the speed of light as an invariant. It has been argued that there is a sound physical basis for such a modification. However, up to now, DSR exists as a project rather than a complete theory. In particular, there is a controversy concerning the many-particle sector. In spite of that the following assumptions seem to be common to all versions of DSR theories: (i) the momentum space is commutative; (ii) the symmetry group *is* the Lorentz group; however, it can act in the momentum space nonlinearly; (iii) the rotation subgroup acts in the standard way; i.e., the symmetry linearizes on the rotation subgroup and the four-momentum transforms as a triplet plus singlet.

In the present paper we intend to touch on one point concerning the definition of velocity in DSR. There has been some discussion in the literature concerning the proper definition of velocity [19] and there even have been claims [21] that this problem points to a serious obstacle of the theory. This is in spite of the fact that the satisfying solution has been already given more or less explicitly [6,15]. Below we discuss this problem in more general terms showing that there exists a consistent definition which can be applied in all versions of DSR theories respecting the assumptions (i)–(iii). This is done in Sec. II. Section III is devoted to some conclusions; in particular, we comment on recent papers concerning the velocity problem.

II. VELOCITY IN DSR THEORIES

The main obvious demand is that the velocity is a property of reference frame rather than of a particular object. In order to satisfy this demand we can make use of the fact that the group structure of space-time transformations remains the same as in Einstein's theory; then, velocity can be viewed as a parameter of Lorentz transformations.

We shall consider here the massive case, i.e., we assume that the set of allowed values of the four-momentum contains the nontrivial element invariant under rotations. In order to characterize the action of the Lorentz group in momentum space let us recall that the classification of nonlinear realizations of groups is well known [29]. Strictly speaking the theory presented in [29] applies to the compact group case. However, it can be adopted in the massive case considered here. This is because in the massive case there exists a point in the momentum space which is invariant under the action of the compact subgroup (rotations) and we are looking for realizations linearizing on this subgroup. The main conclusion coming from Ref. [29] is that there is basically one realization parametrized by the elements Λ^i_0 of the Lorentz group; it reduces to the triplet on the rotation subgroup. By adding a singlet one can linearize the action of the Lorentz group; the additional constraint, necessary in order to keep the same number of independent coordinates, is simply the mass-shell condition. All other realizations are obtained by a, in general nonlinear, change of variables. Therefore, the final conclusion is that a nonlinear action of the Lorentz group on momentum space,

$$p'^{\mu}=f^{\mu}(p;\Lambda),$$
$$f^{\mu}(f(p;\Lambda),\Lambda')=f^{\mu}(p;\Lambda'\Lambda) \tag{1}$$

has a form

$$f^{\mu}(p;\Lambda)=F^{\mu}(\Lambda F^{-1}(p)), \tag{2}$$

where

$$q^{\mu}\rightarrow p^{\mu}=F^{\mu}(q) \tag{3}$$

is some mapping from the standard four-momentum space to the actual one. In general, this mapping transforms invertibly part of the standard energy-momentum space (for example, the interior of light cone) onto some sector of the modified space. This is because we want to keep some finite energy/momentum Lorentz invariant while for the standard action of the Lorentz group only infinity of the energy-momentum space is such an invariant.

In the standard case the mass shell Σ_m is isomorphic to the coset space SO(3,1)/SO(3) and SO(3) is a stability subgroup of $r=(m,\vec{0})$. Due to the fact that the map F preserves the linear action of SO(3), $k^{\mu}=F^{\mu}(r)$ takes the same form, $k=(M,\vec{0})$. Let us call $\tilde{\Sigma}_M$ the image of Σ_m under F. Any four-momentum p belonging to $\tilde{\Sigma}_M$ can be obtained from k by the action of some element of the Lorentz group,

$$p^{\mu}=f^{\mu}(k;\Lambda). \tag{4}$$

In order to define Λ in unique way we demand it to be a boost,

$$\Lambda=B(\vec{v}) \tag{5}$$

corresponding to the velocity $\vec{v}$; the uniqueness follows from the fact that two Lorentz transformations giving rise to the same p must differ by an element of the stability subgroup of k, i.e., the rotation. Let us note that the decomposition of an arbitrary element Λ of the Lorentz group into the boots and rotation reads

$$\Lambda = B\cdot R, \tag{6}$$

where

$$B^{\mu}{}_{\nu} = \begin{cases} \Lambda^{\mu}{}_{0}, & \nu=0 \\ -\Lambda_{i}{}^{0}, & \mu=0,\ \nu=i \\ \delta^{i}_{j} - \dfrac{\Lambda^{i}{}_{0}\Lambda_{j}{}^{0}}{1+\Lambda^{0}{}_{0}}, & \mu=i,\ \nu=j \end{cases} \tag{7}$$

and

$$R^{\mu}{}_{\nu} = \begin{cases} \delta^{\mu}{}_{\nu}, & \mu=0 \ \text{or}\ \nu=0 \\ \Lambda^{i}{}_{j} - \dfrac{\Lambda^{i}{}_{0}\Lambda^{0}{}_{j}}{1+\Lambda^{0}{}_{0}}, & \mu=i,\ \nu=j. \end{cases} \tag{8}$$

Thus, one arrives at the following relation:

$$p^{\mu} = f^{\mu}(k;B(\vec{v})). \tag{9}$$

We define $\vec{v}$ to be the velocity of the particle characterized by the four-momentum p. It is easily seen that this definition gives the proper addition law; indeed

$$\begin{aligned} p'^{\mu} &= f^{\mu}(p;B(\vec{v}\,')) = f^{\mu}(f(k;B(\vec{v}));B(\vec{v}\,')) \\ &= f^{\mu}(k;B(\vec{v}\,')B(\vec{v})) = f^{\mu}(k;B(\vec{v}\oplus\vec{v})R) \\ &= f^{\mu}(k;B(\vec{v}\,'\oplus\vec{v})), \end{aligned} \tag{10}$$

where $\vec{v}\,'\oplus\vec{v}$ denotes the Einstein addition law for velocities.

By specifying the form of F one can give an explicit formula for the relation between the four-momentum p and velocity $\vec{v}$. The general form of F reads

$$\begin{aligned} p^{0} &= F^{0}(q) = \tilde{H}(q^{0},\vec{q}^{\,2}) \equiv H(q^{0},m^{2}), \\ p^{i} &= F^{i}(q) = \tilde{G}(q^{0},\vec{q}^{\,2})q^{i} \equiv G(q^{0},m^{2})q^{i}. \end{aligned} \tag{11}$$

Using Eqs. (2) and (9) one can easily show that

$$\vec{v} = \frac{\partial H^{-1}(p^{0})}{\partial p^{0}} \left(\frac{\partial}{\partial q^{0}}(q^{0}G(q^{0})) - \frac{m^{2}}{q^{0}}\frac{\partial G}{\partial q^{0}} \right)\Bigg|_{q^{0}=H^{-1}(p^{0})} \cdot \frac{\partial p^{0}}{\partial \vec{p}}, \tag{12}$$

where $\partial p^{0}/\partial \vec{p}$ is calculated from the deformed mass-shell condition

$$G^{2}(H^{-1}(p^{0}))(H^{-1}(p^{0}))^{2} - p^{2} = m^{2}G^{2}(H^{-1}(p^{0})). \tag{13}$$

Indeed, keeping in mind that q is the standard four-momentum, we can write

$$\vec{v} = \frac{\partial q^{0}}{\partial \vec{q}}. \tag{14}$$

Now, q^{0} is a function of p^{0} and m^{2}; on the other hand, p^{0} itself can be expressed through p^{2} using the deformed mass-shell condition. Therefore, we obtain

$$\frac{\partial q^{0}}{\partial q^{i}} = \frac{\partial q^{0}}{\partial p^{0}}\frac{\partial p^{0}}{\partial p^{j}}\frac{\partial p^{j}}{\partial q^{i}} = \frac{\partial H^{-1}(p^{0})}{\partial p^{0}}\frac{\partial p^{0}}{\partial p^{j}}\frac{\partial p^{j}}{\partial q^{i}}. \tag{15}$$

Equation (11) implies in turn

$$\frac{\partial p^{j}}{\partial q^{i}} = \frac{\partial G(q^{0})}{\partial q^{0}}\frac{q^{i}q^{j}}{q^{0}} + G(q^{0})\delta_{ij}. \tag{16}$$

Noting that, due to rotational invariance, $\partial p^{0}/\partial\vec{p}$, $\vec{p}$, and $\vec{q}$ are parallel and using once more the mass-shell condition $(q^{0})^{2} = q^{2}+m^{2}$ we arrive at the formula (12). Note that we could also start from the relation $\vec{v} = \vec{q}/q^{0}$ to arrive at the equivalent form

$$\vec{v} = \frac{\vec{p}}{H^{-1}(p^{0})G(H^{-1}(p^{0}))}. \tag{17}$$

However, we prefer to use Eq. (12) to present the evidence that the undeformed Hamiltonian formalism is, in general, not applicable, $\vec{v} \neq \partial p^{0}/\partial\vec{p}$.

Let us apply our formula to some particular cases. First, let us consider the κ-Poincare case. The general form of the mapping (11) has been given in Ref. [30] and reads

$$\begin{aligned} H(q^{0},m^{2}) &= \kappa \ln\left(\frac{q^{0}+C(m^{2})}{C(m^{2})-A(m^{2})} \right), \\ G(q^{0},m^{2}) &= \frac{\kappa}{q^{0}+C(m^{2})}, \end{aligned} \tag{18}$$

where $A(m^{2})$ and $C(m^{2})$ obey

$$A^{2}(m^{2}) - 2A(m^{2})C(m^{2}) + m^{2} = 0. \tag{19}$$

The partial derivative $\partial p^{0}/\partial\vec{p}$ is calculated from the deformed mass-shell relation

$$\frac{2A}{C-A} = \frac{1}{\kappa^{2}}\left[4\kappa^{2} sh^{2}\left(\frac{p^{0}}{2\kappa}\right) - e^{p^{0}/\kappa}p^{2} \right]. \tag{20}$$

Equation (12) implies in this case

$$\vec{v} = \frac{2\vec{p}}{\kappa\left(1 - e^{-2p^{0}/\kappa} + \dfrac{p^{2}}{\kappa^{2}}\right)} \tag{21}$$

which coincides with the right group velocity defined in Ref. [15] (see also [31]).

As a second example consider the Magueijo-Smolin version of DSR [6]. The general form of the mapping (11) is now

$$H(q^0,m^2)=\frac{\kappa q^0}{q^0+C(m^2)},$$

$$G(q^0,m^2)=\frac{\kappa}{q^0+C(m^2)}, \tag{22}$$

while the deformed mass-shell condition reads

$$\frac{(p^0)^2-p^2}{\left(1-\frac{p^0}{\kappa}\right)^2}=\frac{\kappa^2m^2}{C^2(m^2)}. \tag{23}$$

A simple computation gives

$$\vec{v}=\frac{\vec{p}}{p^0} \tag{24}$$

again in accordance with the original result [6].

Let us note that in both cases $\vec{v}\neq\partial p^0/\partial\vec{p}$. This is related to the fact that one should use a deformed Hamiltonian formalism [32] (cf. the remark above).

III. CONCLUSIONS

We have derived the general formula for velocity in any DSR theory obeying (i)–(iii) [Eq. (12)]. It satisfies, by construction, the Einstein addition law; this is a desirable property which makes easier the space-time interpretation of $\vec{v}$. The derivation is based on very general assumptions and our conclusions in most cases agree with those obtained by other authors.

Toller [24] gave independent arguments in favor of the statement that the modified energy-momentum variables arise from the change of coordinates in the four-momentum space and discuss the relation between the four-momentum and velocity.

Mignemi [25,26] investigated the transformation laws of position and momentum in a Hamiltonian setting and analyzed in detail the assumption that the definition of velocity should not depend on the particle mass.

Kimberly *et al.* [27] discussed in some detail the space-time aspect of DSR theories. They propose two approaches; in the first one the position and momentum spaces are essentially independent while in the second the Lorentz transformations become energy dependent. In the latter case the speed of light is also energy dependent and is given by E/p and not dE/dp.

Daszkiewicz *et al.* [23] considered the velocity problem from the point of view of the Hamiltonian formalism. They assumed that the velocity is defined through the Poisson bracket of position with deformed Hamiltonian taking into account the specific phase space structure of DSR theories. They found that the four-velocity transforms as the standard Lorentz four-vector.

An interesting proposal for DSR, based on quantum conformal algebra, has been recently given in [22]. The underlying symmetry is described by deformed Poincaré-Weyl Hopf algebra, the deformation parameter being identified with the Planck length. Within this framework two proposals for the definition of velocity have been given; one of them gives rise to constant speed of light while the second implies variable speed.

ACKNOWLEDGMENTS

This work was supported by KBN grant 5 P03B05620.

[1] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002).
[2] G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001).
[3] G. Amelino-Camelia, D. Benedetti, and F. D'Andrea, hep-th/0201245.
[4] M. Arzano and G. Amelino-Camelio, gr-qc/0207003.
[5] R. Bruno, G. Amelino-Camelia, and J. Kowalski-Glikman, Phys. Lett. B **522**, 133 (2001).
[6] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).
[7] J. Magueijo and L. Smolin, Phys. Rev. D **67**, 044017 (2003).
[8] J. Kowalski-Glikman, Phys. Lett. A **286**, 391 (2001).
[9] J. Kowalski-Glikman, Mod. Phys. Lett. A **17**, 1 (2002).
[10] J. Kowalski-Glikman, Phys. Lett. A **299**, 454 (2002).
[11] J. Kowalski-Glikman and S. Nowak, Phys. Lett. B **539**, 126 (2002).
[12] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. D **12**, 299 (2003).
[13] J. Kowalski-Glikman, Phys. Lett. B **547**, 291 (2002).
[14] J. Lukierski and A. Nowicki, Int. J. Mod. Phys. A **18**, 7 (2003).
[15] J. Lukierski and A. Nowicki, Acta Phys. Pol. B **33**, 2537 (2002).
[16] J. Lukierski and A. Nowicki, Czech. J. Phys. **52**, 1261 (2002).
[17] S. Judes and M. Visser, Phys. Rev. D **68**, 045001 (2003).
[18] D.V. Ahluwalia and M. Kirchbach, gr-qc/0207004.
[19] A. Granik, hep-th/0207113.
[20] N.R. Bruno, Phys. Lett. B **547**, 109 (2002).
[21] J. Rembieliński and K. Smoliński, hep-th/0207031.
[22] A. Ballesteros, N. Bruno, and F. Herranz, hep-th/0305033.
[23] M. Daszkiewicz, K. Imilkowska, and J. Kowalski-Glikman, hep-th/0304027.
[24] M. Toller, hep-th/0301153.
[25] S. Mignemi, hep-th/0302065.
[26] S. Mignemi, Phys. Rev. D (to be published), gr-qc/0304029.
[27] P. Kimberly, J. Magueijo, and J. Medeiros, gr-qc/0303067.
[28] P. McCarthy and R.L. Schafir, gr-qc/0304079.
[29] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239

(1969); C.G. Callan, S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).
[30] S. Giler, C. Gonera, and M. Majewski, Acta Phys. Pol. B **27**, 2131 (1996).
[31] T. Tamaki, T. Harada, U. Miyamoto, and T. Torii, Phys. Rev. D **66**, 105003 (2002).
[32] J. Lukierski, P. Stichel, and W.J. Zakrzewski, Ann. Phys. **260**, 224 (1997).